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Structure of Two-Dimensional Sandpile. I. Height Probabilities

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The height probabilities of the two-dimensional Abelian sandpile model are the fractional numbers of lattice sites having heights 1, 2, 3, 4. A combinatorial method for evaluation of these quantities is proposed. The method is based on mapping the set of allowed sandpile configurations onto the set of spanning trees covering a given lattice. Exact analytical expressions for all probabilities are obtained.

KEY WORDS: Self-organized criticality; sandpiles; spanning trees; height probabilities.

1. INTRODUCTION AND RESULTS

The sandpile model proposed by Bak *et al.*[1] has been the subject of active research. It has been found to be a prototype of such diverse phenomena as earthquakes,[2] luminosity of stars,[3] river flows,[4] coagulation,[5] relaxation phenomena in magnets,[6] and neural networks.[7] Generally, the sandpile model provides a unifying concept for large-scale behavior in dissipative systems with many degrees of freedom, and involves essential properties of self-organized criticality (SOC).

The process of formation of a sandpile is formulated in terms of heights of the pile and toppling conditions. If the toppling at a lattice site depends only on the height at that site, the sandpile model has an Abelian group structure and is analytically tractable. Using the Abelian property of the model, Dhar[8] has determined the total number of allowed configurations of the sandpile in the SOC state. Also, the found the correlation

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function measuring the expected number of topplings at a given site due to a particle added at another one. In spite of apparent progress, the analytical description of the model is far from being complete.

The characterization of the SOC state has two aspects, dynamical and structural. The first one implies a description of avalanches, their duration, mass, linear extent, perimeter, etc. The structural characteristics of the sandpile are fractional numbers of sites having a given height and correlations between heights at different sites in a typical allowed configuration. In this paper, we deal with the structure of the 2D sandpile model. The first part of the paper is devoted to a combinatorial treatment and exact evaluation of the height probabilities. A short account of some of these results has been published previously.[9]

The height of a sandpile at any site of a 2D square lattice takes values 1, 2, 3, 4 in the SOC state. The first numerical estimation of probabilities $P(1)$, $P(2)$, $P(3)$, and $P(4)$ was made by Zhang[10] for a model with continuous heights: $P(1)=0.10$, $P(2)=0.16$, $P(3)=0.32$, $P(4)=0.42$. The corresponding data for the discrete sandpile model on the lattice of linear sizes 30, 40 were obtained by Erzan and Sinha[11]: $P(1)=0.07\pm 4\%$, $P(2)=0.17\pm 7\%$, $P(3)=0.31\pm 9\%$, $P(4)=0.45\pm 3\%$. Extensive simulations for the lattice of size 672 were undertaken by Manna,[12] who found $P(1)=0.0736$, $P(2)=0.174$, $P(3)=0.307$; $P(4)=0.446$, with typical errors of the order of 0.003. Grassberger and Manna[13] performed simulations on some even larger lattices, but not with sufficiently high statistics. Their results for the lattice of size 672 are $P(1)=0.0736$, $P(2)=0.1740$, $P(3)=0.3062$, $P(4)=0.4462$.

The first exact result for the height probability $P(1)$ was obtained by Majumdar and Dhar.[14] They found

$$P(1)=\frac{2}{\pi^2}-\frac{4}{\pi^3}=0.07363... \tag{1}$$

The problem of finding $P(2)$, $P(3)$, and $P(4)$ turned out more difficult due to clusters of growing size giving a contribution to these probabilities. Attempts at analytical determination of $P(2)$ showed a very slow convergence of cluster series and gave only the lower bound[14]

$$P(2)\geqslant 0.131438 \tag{2}$$

In this paper, we present a method giving the exact solution of the problem in two dimensions. We derive analytical expressions for $P(2)$, $P(3)$, and $P(4)$ which read in the limit of an infinitely large lattice

$$P(2)=\frac{1}{2}-\frac{3}{2\pi}-\frac{2}{\pi^2}+\frac{12}{\pi^3}+\frac{I_1}{4} \tag{3}$$

$$P(3)=\frac{1}{4}+\frac{3}{2\pi}+\frac{1}{\pi^2}-\frac{12}{\pi^3}-\frac{I_1}{2}-\frac{3I_2}{32} \tag{4}$$

$$P(4)=\frac{1}{4}-\frac{1}{\pi^2}+\frac{4}{\pi^3}+\frac{I_1}{4}+\frac{3I_2}{32} \tag{5}$$

Here I_ν, $\nu=1, 2$, are integrals:

$$I_\nu=\frac{1}{(2\pi)^4}\iiiint_0^{2\pi}\frac{i\sin(\beta_1)\det(M_\nu)}{D(\alpha_1,\beta_1)\,D(\alpha_2,\beta_2)\,D(\alpha_1+\alpha_2,\beta_1+\beta_2)}\,d\alpha_1\,d\alpha_2\,d\beta_1\,d\beta_2 \tag{6}$$

where

$$D(\alpha,\beta)=2-\cos(\alpha)-\cos(\beta) \tag{7}$$

and M_1, M_2 are matrices,

$$M_1=\begin{pmatrix} 1 & 1 & e^{i\alpha_2} & 1 \\ 3 & e^{i(\beta_1+\beta_2)} & e^{i(\alpha_2-\beta_2)} & e^{-i\beta_1} \\ 4/\pi-1 & e^{i(\alpha_1+\alpha_2)} & 1 & e^{-i\alpha_1} \\ 4/\pi-1 & e^{-i(\alpha_1+\alpha_2)} & e^{2i\alpha_2} & e^{i\alpha_1} \end{pmatrix} \tag{8}$$

and

$$M_2=\begin{pmatrix} e^{i\beta_2} & e^{-i(\alpha_1+\alpha_2)-i(\beta_1+\beta_2)} & e^{i\beta_1} \\ e^{-i\alpha_2} & 1 & e^{-i\alpha_1} \\ e^{i\alpha_2} & e^{-2i(\alpha_1+\alpha_2)} & e^{i\alpha_1} \end{pmatrix} \tag{9}$$

The numerical evaluation of integrals in Eq. (6) leads to $P(2)=0.1739...$, $P(3)=0.3063...$, $P(4)=0.4461...$, in good agreement with the high-statistics data.

The solution is based on mapping the set of allowed sandpile configurations onto the set of spanning trees covering a given lattice. The local characteristics of the sandpile $P(i)$, $i=2, 3, 4$, can be related to nonlocal characteristics of trees, namely to probabilities of branches obeying some special conditions. To take into account these conditions, we use the equivalent formulation of the spanning tree model in the language of acyclic arrow configurations. The resulting representation of $P(i)$ in terms of arrow correlation functions permits one to evaluate the height probabilities by using the modified Kirchhoff theorem.

2. GENERAL PROPERTIES OF SANDPILE CONFIGURATIONS

We consider a large square lattice L consisting of n sites. The sandpile is characterized by integer heights z_i at all sites i and is specified by two rules:

(i) Adding a particle at a random site: $z_i \to z_i + 1$.

(ii) The toppling rule: if any $z_i > 4$, then $z_i \to z_i - 4$ and $z_j \to z_j + 1$, $|i - j| = 1$.

Particles can leave the system at the edges where the number of neighbor sites is less than 4, the number of toppled particles.

The sand pile model is a cellular automaton. To describe a Markovian evolution of the model, it is convenient to introduce[8] operators a_i ($i = 1, \ldots, n$) on the space of stable configurations by requiring $a_i C$ be a stable configuration obtained by adding a particle at a site i to the configuration C and allowing the system to evolve by toppling. It has been argued[8] that operators a_i commute with each other

$$[a_i, a_j] = 0 \tag{10}$$

for all i, j. The steady state of the model is represented by recurrent configurations obeying the property

$$a_i^{m_i} C = C \tag{11}$$

for all i, where m_i are positive integers. The properties Eqs. (10) and (11) permit one to conclude that the invariant state of the sandpile evolution (the SOC state) has a rather simple structure: (i) only recurrent configurations have a nonzero probability; (ii) the probabilities of all recurrent configurations are equal.

The total number of stable configurations of the sandpile is 4^n. Some of them are forbidden in the SOC state. Following Dhar,[8] we define a forbidden subconfiguration (FSC) as any subset $F \subset L$ of lattice sites if the corresponding heights z_j, $j \in F$, satisfy the inequalities $z_j \leqslant$ coordination number of j in F. A configuration that does not contain FSC is called an allowed configuration. All recurrent configurations are allowed.[8] The converse statement has been proved in ref. 15. Therefore, we can formulate our problem as follows: for a square lattice with given boundary conditions it is necessary to find the average fractional numbers of heights 1, 2, 3, 4 at the set of all allowed configurations.

Dhar[8] has proposed a recursive procedure called the burning algorithm, to determine if a given configuration is allowed. One deletes step by step from a given configuration any site j whose height z_j is greater than the coordination number of j in a lattice resulting after a preceding step. If in the end the lattice becomes empty, the configuration is allowed. The number of stable allowed configurations is given by the remarkably simple formula[8]

$$N = \det \Delta \tag{12}$$

where Δ is an $n \times n$ discrete Laplacian matrix with $\Delta_{ij} = 4$ if $i = j$, $\Delta_{ij} = -1$ if $|i-j| = 1$, and $\Delta_{ij} = 0$ otherwise.

For a given lattice site i_0, the set of allowed configurations can be divided into four subsets s_1, s_2, s_3, s_4. These are defined as follows. A configuration C belongs: to a subset s_1 if it remains allowed after all substitutions $z_0 = 1, 2, 3, 4$ at i_0; to subset s_2 if it remains allowed for $z_0 = 2, 3, 4$ and becomes forbidden for $z_0 = 1$; to subset s_3 if it remains allowed for $z_0 = 3, 4$ and becomes forbidden for $z_0 = 1, 2$. The subset s_4 contains configurations which are allowed only for $z_0 = 4$. All admitted substitutions at a given site correspond to equal numbers of configurations. Therefore, the height probabilities $P(1)$, $P(2)$, $P(3)$, and $P(4)$ can be written in the form

$$P(1) = \frac{N_1}{4N} \tag{13}$$

$$P(2) = P(1) + \frac{N_2}{3N} \tag{14}$$

$$P(3) = P(2) + \frac{N_3}{2N} \tag{15}$$

$$P(4) = P(3) + \frac{N_4}{N} \tag{16}$$

where N_i is the number of allowed configurations in subsets s_i, $i = 1, 2, 3, 4$.

3. DESCRIPTION OF SETS s_1, s_2, s_3, s_4

The description of s_1 is given in ref. 14. If a configuration C is allowed for $z_0 = 1$, it remains allowed after substitutions $z_0 = 2, 3, 4$. Due to Eq. (13), N_1 is equal to the number of allowed configurations with $z_0 = 1$ multiplied by 4. Let us fix $z_0 = 1$. For every allowed configuration, there exists a burning procedure b_1, b_2, b_3,... which burns the site i_0 after all its nearest neighbors j_1, j_2, j_3, j_4. Therefore, the number of allowed configurations with $z_0 = 1$ does not depend on the existence of the site i_0 and one can delete it from the lattice L together with adjacent bonds. Alternatively, one may remove three bonds, say $i_0 j_2$, $i_0 j_3$, $i_0 j_4$ from L, reducing simultaneously diagonal elements $\Delta_{i_0 i_0}$ by 3 and $\Delta_{j_2 j_2}$, $\Delta_{j_3 j_3}$, and $\Delta_{j_4 j_4}$ by 1. Each allowed configuration on the new lattice L' with the new matrix Δ' corresponds to an allowed configuration on L with $z_0 = 1$. Applying the determinant formula (12) to the lattice L', we get

$$N_1 = 4 \det \Delta' \tag{17}$$

Let us now look at Eqs. (12) and (17) from a different point of view. First, we recall some definitions of graph theory. A subgraph G of the graph L is a subset of vertices and bonds of L such that it forms a graph. Denote by $\nu(G)$, $\mu(G)$, and $\kappa(G)$ the numbers of vertices, connected parts, and internal loops of G, respectively. A subgraph T is a spanning tree of L if $\nu(T) = \nu(L)$, $\mu(T) = 1$, and $\kappa(T) = 0$. The coordination number $\deg(i)$ of a site of a tree is the number of edges meeting at that site.

To simplify the consideration, we specify the boundary conditions as follows: $\Delta_{ii} = 3$ if i belongs to the edge of L, $\Delta_{ii} = 2$ if i belongs to one of three corners, and $\Delta_{ii} = 3$ if i coincides with the fourth corner denoted by ★.

According to the Kirchhoff theorem,[16] $\det \Delta$ is the number of spanning trees of the lattice L. By construction, $\det \Delta'$ is the number of spanning trees T' satisfying the following conditions:

(a) Each T' contains the bond $i_0 j_1$.

(b) The coordination number $\deg(i_0) = 1$, $i_0 \in T'$.

Using the symmetry of the lattice in the thermodynamic limit, we get the following rule for determination N_1: N_1 is the number of spanning trees of the lattice L having $\deg(i_0) = 1$.

Finally, we introduce the ordering of the lattice with respect to the root ★. We shall say that a site i is the predecessor of a site j if the unique path to the root ★ along the tree from i goes through j. For the trees contributing to N_1 there are no predecessors of i_0 among neighbor sites j_1, j_2, j_3, j_4. So we obtain the third definition of N_1:

$$N_1 = X_0 \tag{18}$$

where X_0 is the number of spanning trees for which sites j_1, j_2, j_3, j_4 are not predecessors of i_0 (Fig. 1).

Let us turn to the description of s_2. By definition, the substitution $z_0 = 1$ converts an arbitrary configuration $C \in s_2$ into a forbidden one C'. This means that a FSC appears which contains the site i_0 with $z_0 = 1$, one of the sites j_1, j_2, j_3, j_4, say j_1, with $z_{j_1} \geqslant 1$, and some k connected sites ($k \geqslant 0$) including none of the sites j_2, j_3, j_4. (If one of j_2, j_3, j_4 also

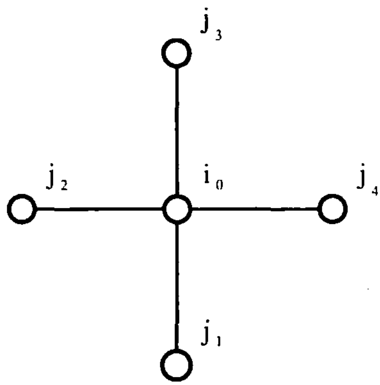

Fig. 1. Diagram representation of s_1. Open circles are the sites which are not predecessors of i_0.

belongs to the FSC, then the configuration C' remains forbidden after the substitution $z_0 = 2$.)

Let $S(C)$ be the FSC resulting from the substitution $z_0 = 1$ in C. We construct a lattice L' in the following way. We delete the boundary bonds connecting the sites of $S(C)$ to the rest of the lattice L with the exception of the only bond connecting the site i_0 with one of the sites j_2, j_3, j_4 (j_2 for definiteness). For each bond deleted, we also decrease the maximum height allowed at the two end sites of the bond by 1. In this way, we obtain a new toppling rule matrix $\Delta'(S)$ which depends on the form of a given FSC. As above, for each allowed configuration C with $z_0 = 2$ a burning procedure exists which does not depend on the presence of deleted bonds. Therefore, the set of all allowed configurations on the lattice L' is in one-to-one correspondence to the set of configurations $C \in s_2$ which generates S by the substitution $z_0 = 1$. As the sites j_1, j_2, j_3, j_4 are equivalent and three possibilities $z_0 = 2, 3, 4$ contribute to s_2, the number of allowed configurations in s_2 is

$$N_2 = 12 \sum_S \det \Delta'(S) \tag{19}$$

where the sum runs over all possible FSCs containing the sites i_0, j_1 and none of the sites j_2, j_3, j_4. Using the Kirchhoff theorem, we conclude that the sum in Eq. (19) is the number of spanning trees T' satisfying the following conditions:

(a) Each T' contains bonds $j_1 i_0$ and $i_0 j_2$.

(b) Deletion of the bond $i_0 j_2$ divides T' into two subtrees T_1 and T_2 such that the sites i_0 and j_1 belong to T_1 and the sites ★, j_2, j_3, j_4 belong to T_2.

(c) The bonds $i_0 j_3$ and $i_0 j_4$ are always absent among the bonds of T'.

The rules (a)–(c) imply that the site j_1 is the nearest predecessor of i_0; all other sites of $S(C)$ are also predecessors of i_0; none of the sites j_2, j_3, j_4 are predecessors of i_0 (Fig. 2). Summarizing, we can formulate the new rule for the determination of N_2:

$$N_2 = X_1 \tag{20}$$

where X_1 is the number of all spanning trees for which the site i_0 has only one predecessor among its nearest neighbor sites.

The description of s_3 is quite similar to that of s_2. The substitution $z_0 = 2$ produces FSCs which contain the site i_0 with $z_0 = 2$, two nearest neighbor sites, and some k sites ($k \geqslant 0$) belonging to them. In contrast with the previous case, one of the neighbor sites belonging to FSC is not necessarily a nearest predecessor of i_0 and may be connected with i_0 via an arbitrary long sequence of bonds along a tree (Fig. 3b). Omitting the construction of new toppling matrices, we can write

$$N_3 = X_2^{(1)} + X_2^{(2)} + X_2^{(3)} \tag{21}$$

where $X_2^{(i)}$, $i = 1, 2, 3$, are the numbers of spanning trees for which positions of two predecessors with respect to i_0 are shown in Fig. 3a–3c.

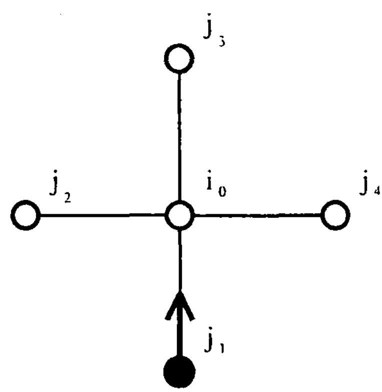

Fig. 2. Diagram representation of s_2. The arrow at the closed circle indicates that j_1 is the nearest predecessor of i_0.

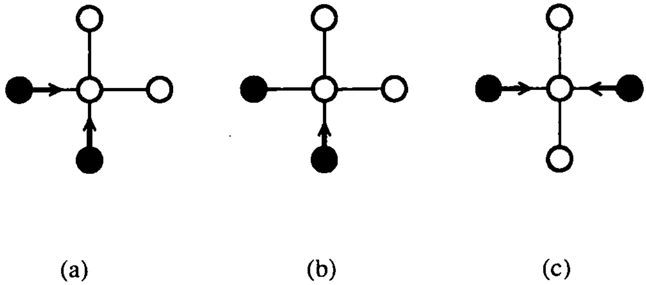

Fig. 3. Diagram representation of s_3. The left closed circle in case (b) denotes a predecessor of i_0. The rest of the closed circles provided with arrows are nearest predecessors of i_0.

The description of s_4 is clear from Fig. 4. The site i_0 is surrounded by three predecessors. In addition to the previous cases we must distinguish two possibilities for a predecessor to be connected with the nearest predecessor (Figs. 4b and 4d). Counting six configurations of neighbors around i_0, we have

$$N_4 = X_3^{(1)} + X_3^{(2)} + \cdots + X_3^{(6)} \tag{22}$$

where $X_3^{(i)}$, $i = 1,\dots, 6$, is the number of spanning trees obeying the given condition.

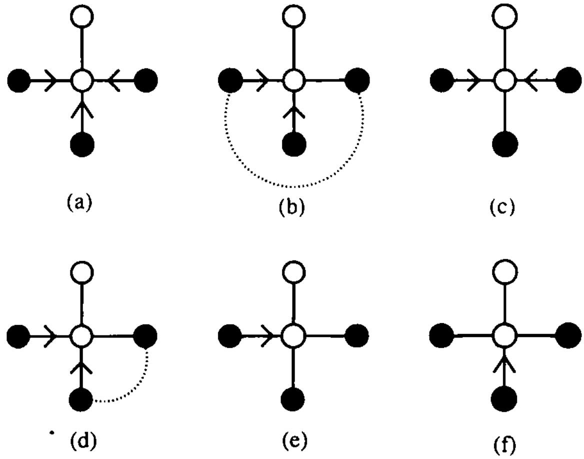

Fig. 4. Diagram representation of s_4. Broken lines denote different paths along a tree connecting predecessors with nearest predecessors. The position of the sites j_1, j_2, j_3, j_4 with respect to i_0 coincide with those in Fig. 1.

4. ARROW CONFIGURATIONS

It is convenient to introduce a different description of a tree configuration. Let each lattice site i except ★ contain an arrow which can be directed from i to one of its nearest neighbors i'. We say that an arrow generates a path ii' from i to i'. A collection of paths of the form $i_1 i_2$, $i_2 i_3,..., i_3,..., i_{k-1} i_k$ is a path $i_1 i_k$ from i_1 to i_k. If the site i_k coincides with i_1, the path $i_1 i_k$ is closed. If a configuration of arrows generates no closed paths, it is an acyclic one.

The acyclic configurations are in one-to-one correspondence to the spanning trees of a given lattice. Indeed, let us ascribe to each vertex i of the tree an arrow directed from i to the nearest neighbor i' for which a distance (the number of connected bonds) between i' and ★ is minimal. We get a configuration of arrows which generates no closed paths. Conversely, consider an arrow configuration. The absence of closed paths implies that each generated path ends at the site ★. Then a collection of bonds belonging to all paths forms a spanning tree having the root ★.

The diagram representation allows one to redefine the sets s_1, s_2, s_3, s_4 in the arrow language. A part of neighbor sites of i_0 in Figs. 1–4 is already marked by arrows. The arrows at unmarked sites j_1, j_2, j_3, j_4 may be directed anywhere, but not to i_0. An arrow at i_0 may be directed to any open circle. Enumeration of arrow configurations with fixed positions of arrows at given sites can be easily performed by using the Kirchhoff theorem. To this end, it is enough to replace by zero the matrix elements Δ_{ij} corresponding to forbidden directions of arrows and to evaluate the determinant of the resulting matrix Δ'. For example,

$$X_0 = \det \Delta' \tag{23}$$

where Δ' differs from Δ by elements $\Delta'_{j_1 i_0} = \Delta'_{j_2 i_0} = \Delta'_{j_3 i_0} = \Delta'_{j_4 i_0} = 0$ and $\Delta'_{j_1 j_1} = \Delta'_{j_2 j_2} = \Delta'_{j_3 j_3} = \Delta'_{j_4 j_4} = 3$.

Evaluation of $\det \Delta'$ is straightforward due to the formula

$$\frac{\det \Delta'}{\det \Delta} = \det(I + G\delta) \tag{24}$$

where $\delta = \Delta' - \Delta$ and the matrix $G = \Delta^{-1}$ is the two-dimensional lattice Green function given by

$$G(r, r) - G(r, r') = \frac{1}{2(2\pi)^2} \int_0^{2\pi} \int \frac{1 - \cos[(x - x')\alpha] \cos[(y - y')\beta]}{2 - \cos(\alpha) - \cos(\beta)} \, d\alpha \, d\beta \tag{25}$$

Hereafter, we shall use also the notation

$$g_{k,l} = G(r, r'); \qquad r = (x, y), \quad r' = (x+k, y+l) \tag{26}$$

to display the dependence on relative two-dimensional coordinates.

Crossing out those rows and columns of δ containing only zero elements, and using tabulated values of the Green functions,[17] we obtain

$$\frac{X_0}{N} = \frac{8}{\pi^2} - \frac{16}{\pi^3} \tag{27}$$

Due to Eqs. (13) and (18) one gets

$$P(1) = \frac{2}{\pi^2} - \frac{4}{\pi^3} \tag{28}$$

This is the value obtained by Majumdar and Dhar.[14] Evaluation of diagrams responsible for $P(2)$, $P(3)$, and $P(4)$ goes beyond the validity of the Kirchhoff theorem. Indeed, the diagrams in Fig. 2–4 involve a nonlocal condition for a given site not to be a predecessor of another. Therefore, we cannot simply fix positions of a finite number of arrows to reproduce a situation around i_0. We can try, however, to find several relations between nonlocal diagrams using only local conditions. Two of them are shown in Fig. 5.

Numbers R_1 and R_2, like X_0, are determinants of perturbed matrices Δ'. In the case of R_1, the matrix Δ' contains zero elements $\Delta'_{j_3 i_0} = \Delta'_{j_4 i_0} = 0$

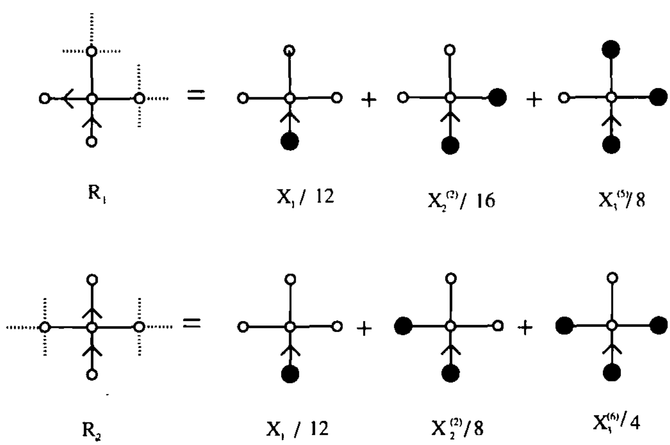

Fig. 5. Correspondence between local arrow configurations R_1, R_2 and diagrams involving nonlocal conditions. The dotted lines mark three possible positions of an arrow at a given site.

corresponding to forbidden directions of arrows at j_3 and j_4. Besides, the elements connecting the sites i_0 and j_1 with their nearest neighbors equal zero except $\Delta'_{j_1 i_0} = \Delta'_{i_0 j_2} = -1$. In the case of R_2 the forbidden and allowed directions are defined by $\Delta'_{j_2 i_0} = \Delta'_{j_4 i_0} = 0$ and $\Delta'_{j_1 i_0} = \Delta'_{i_0 j_3} = -1$, respectively. Each diagonal element Δ'_{ii} in both cases is equal to the number of allowed directions of arrows at the site i. Evaluating the determinants of these matrices, we have

$$\frac{R_1}{N} = \frac{1}{2\pi} - \frac{5}{2\pi^2} + \frac{4}{\pi^3} \tag{29}$$

$$\frac{R_2}{N} = \frac{1}{\pi} - \frac{4}{\pi^2} + \frac{4}{\pi^3} \tag{30}$$

We can also notice equivalence of the two diagrams in Figs. 4e and 4f. Let us reverse directions of all arrows on each path from j_2 to j_1 in Fig. 4f. Simultaneously, we replace the arrow at j_1 pointing to i_0 by one at j_2. As a result, we get the configuration shown in Fig. 4e. Taking into account the symmetry of the diagram of Fig. 4f, we obtain

$$\tfrac{1}{2} X_3^{(5)} = X_3^{(6)} \tag{31}$$

Using Eqs. (29) and (30) and the relationships between R_1, R_2 and X_1, X_2, X_3 shown in Fig. 5, we find $X_2^{(2)}$ in the form

$$\frac{X_2^{(2)}}{N} = \frac{8}{N}(R_2 - R_1) = \frac{4}{\pi} - \frac{12}{\pi^2} \tag{32}$$

The diagram in Fig. 3b is the only nonlocal diagram which can be evaluated by a direct application of the Kirchhoff theorem. The rest of the diagrams need a more elaborate technique. It follows from Eqs. (20) and (21) that it is sufficient to find X_1, $X_2^{(1)}$, $X_2^{(3)}$ besides $X_2^{(2)}$ to determine $P(2)$, $P(3)$, $P(4)$. Let us try to formulate the above-mentioned nonlocal conditions using the arrow notation.

We start with the diagram in Fig. 2. There are three equivalent possibilities to direct the arrow at i_0 to sites j_2, j_3, j_4. Let us choose j_2. Then the condition for the site j_2 not to be the predecessor of i_0 is fulfilled automatically due to the acyclic property of arrow configurations. To take into account the analogous condition for the site j_4, we put one more arrow at i_0 directed to j_4 and demand that the new configuration of arrows on the lattice should also be acyclic, i.e., it does not generate any closed path. If the sites j_2 and j_4 are not predecessors of i_0, the site j_3 also is not the predecessor, because in two dimensions any path from j_3 to ★ is

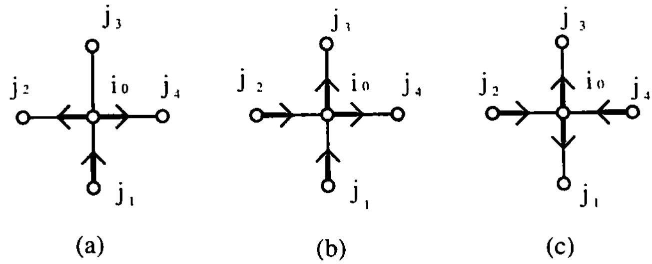

Fig. 6. Acyclic configurations corresponding to (a) X_1, (b) $X_2^{(1)}$, (c) $X_2^{(3)}$.

enclosed between two paths $j_2\bigstar$ and $j_4\bigstar$. In a similar way, we put two arrows at i_0 directed to open circles in Figs. 3a and 3c, ensuring the condition for the sites j_3, j_4 and j_1, j_3 not to be predecessors of i_0. The resulting combinations of arrows are shown in Fig. 6.

The new acyclic configurations containing a site with two arrows at i_0 no longer represent spanning trees, because they involve a loop created by two paths starting at i_0 and ending at $\bigstar$.

5. COMBINATORIAL CONTENT OF KIRCHHOFF THEOREM

Our aim in this section is to construct an analog of the Kirchhoff theorem which would be suitable for enumeration of acyclic arrow configurations containing two arrows at a selected site. To introduce the necessary improvements, we shall consider the combinatorial content of this theorem.

For a given connected graph G consisting of n sites, let Δ_G be an $n \times n$ matrix with elements $\Delta_G(i, j)$, $i, j \in G$:

$$\Delta_G(i, j) = \begin{cases} y_i & \text{if} \quad i = j \\ -x_{ij} & \text{if} \quad \text{sites } i \text{ and } j \text{ are connected by bond} \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

Let $\{x\}$ and $\{y\}$ denote the sets of weights of all bonds and all sites, respectively. It is easy to show[18,19] that the function

$$g(\{x\}, \{y\}) = \det \Delta_G \tag{34}$$

is the generating function of all possible configurations of closed paths weighted in such a way that each bond passed in the direction from i to j gives the weight x_{ij}. Each path brings also a minus sign. The paths have no self-intersections and any two paths have no common lattice site. A site i

not belonging to any path has the weight y_i. For the sake of generality the weights x_{ij} and x_{ji} will be considered not necessarily equal. If one of them, say x_{ji}, is equal to zero, paths passing the bond ij in the direction from j to i give no contribution to the generating function.

The identity (34) follows from the expansion of the determinant into cyclic permutations and underlies practically the solutions of lattice problems belonging to the free-fermion class such as the Ising model and the dimer problem (see ref. 19 for details). Making use Eq. (34), we can prove the Kirchhoff theorem by means of the well-known combinatorial inclusion–exclusion principle.

Let there be N_0 elements and a certain number of properties $p(1)$, $p(2),..., p(n)$. Let, further, N_i be the number of elements with property $p(i)$, and generally let $N_{i_1, i_2,..., i_r}$ be the number of elements with properties $p(i_1)$, $p(i_2),..., p(i_r)$. Then the number of elements N not possessing any of these properties is given by the equation

$$\begin{aligned} N = N_0 - \sum_i N_i + \sum_{i_1 < i_2} N_{i_1, i_2} - \cdots + (-1)^r \\ \times \sum_{i_1 < i_2 < \cdots < i_r} N_{i_1, i_2, ..., i_r} + (-1)^n N_{1, 2, ..., n} \end{aligned} \tag{35}$$

We chose an arbitrary site of G as the root, denoting it by $\bigstar$. Furthermore, we put $y_i = \deg(i)$ for all $i \in G$ except $\bigstar$, for which $y_\bigstar = 1$. In addition, we put $x_{ij} = 1$ for all i, j connected by bonds, except $i = \bigstar$, for which $x_{\bigstar j} = 0$. Let us consider the function $g(\{x\}, \{y\})$ term by term as the sum over all sets of closed paths. The first term arises when the set is empty and equals

$$\prod_{i \neq \bigstar} \deg(i)$$

This term corresponds to a free arrangement of arrows at each of n sites of the graph except $\bigstar$. We shall identify this term with N_0 in Eq. (35). The properties $p(1)$, $p(2),...,$ will be assumed to be closed paths enumerated in an arbitrary order. Then, the second term in Eq. (35) corresponds to the sum over all possible closed paths avoiding $\bigstar$ and taken with a minus sign. Continuing these arguments, we obtain a full correspondence between the function $g(\{x\}, \{y\})$ and the right-hand side of Eq. (35) finding the number of all acyclic arrow configurations or, equivalently, the number of all spanning trees of the graph G.

It should be noted that the elimination of the root plays the same role as evaluating a minor instead of the determinant in the original formulation of the Kirchhoff theorem.[16] An alternative way of taking into account the absence of arrows at the root is to put $y_\star = \deg(\star) + 1$ and $x_{\star j} = 1$ for all j connected with the root by bonds. Indeed, if $x_{\star j} = 1$, then $g(\{x\}, \{y\})$ is the generating function of all possible configurations of closed paths, including those passing via the root. There are $\deg(\star)$ possibilities to arrange an arrow at the root. Each of them creates a closed path starting and ending at the root due to lack of another endpoints. Having opposite signs, these new configurations will be canceled. Therefore, the only possibility of absence of arrows corresponding to the unity in the expression $y_\star = \deg(\star) + 1$ gives a contribution to the final result.

In the case under consideration the graph G is the square lattice with $\deg(i) = 4$. In some cases we shall need also to consider lattices with additional links or forbidden directions of arrows. All these cases can be treated uniformly by putting $x_{ij} = 1$ for allowed directions and $x_{ij} = 0$ for forbidden ones. Generally, putting

$$y_i = \sum_j x_{ij}; \qquad i \neq \star; \quad y_\star = 1; \quad x_{\star j} = 0$$

one obtains the generating function of all acyclic arrow configurations with the weights x_{ij} ascribed to the arrows pointing from i to j.

If a given site contains two fixed arrows, using the inclusion–exclusion principle becomes more complicated. In contrast with the standard acyclic situation, configurations of arrows may appear which generate two closed paths having common sites. In Fig. 7 these are a path P_1 of the type $i_0 j_2 \cdots j_1 i_0$ and a path P_2 of the type $i_0 j_4 \cdots j_1 i_0$ having common sites along the path from i_1 to i_0. Because of the acyclic condition, we should

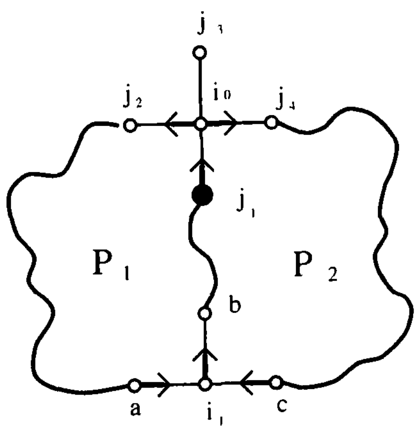

Fig. 7. The configurations of arrows responsible for the Θ-graph. The wavy lines denote all possible paths through the lattice.

provide cancellation both of P_1 and P_2. Then, as configurations containing P_1 and P_2 simultaneously will be excluded twice, we must, according to the inclusion–exclusion principle, return these into the expansion. But the generating function (34) contains only closed nonintersecting paths and we lose the correspondence between Eq. (34) and the expansion (35). To restore this correspondence, we should introduce into the generating function the configurations of arrows forming closed paths of the types P_1 and P_2 simultaneously.

6. HEIGHT PROBABILITIES

In this section, we realize the program outlined in the last two sections. We shall consider evaluation of $P(2)$ in more detail, as it involves all the main steps of the general solution.

We shall denote the configuration of arrows at i_0, j_1, j_2, j_3, j_4 corresponding to X_1 by C_1. The first step is cancellation of the closed paths of type P_1 (Fig. 7) passing via bonds $i_0 j_2$ and $j_1 i_0$. We introduce the matrix $\Delta^{(1)} = \Delta + \delta_{(1)}$ in such a way that positions of arrows on these bonds would be fixed. The following matrix elements $[i, j]$ of $\Delta^{(1)}$ equal zero: $[i_0, j']$, where j' is any neighbor site of i_0 except j_2; $[j_1, j'']$, where j'' is any n.n. site of j_1 except i_0 and also elements $[j_3, i_0]$ and $[j_4, i_0]$. As above, we reduce also the corresponding diagonal elements putting $[i_0, i_0] = [j_1, j_1] = 1$ and $[j_3, j_3] = [j_4, j_4] = 3$. According to the Kirchhoff theorem, $\det \Delta^{(1)}$ enumerates all possible configurations of arrows containing the subconfiguration C_1 except the arrow directed from i_0 to j_4 and generating no closed paths including P_1. Substitution of the matrix $\Delta^{(1)}$ into Eq. (24) gives

$$\frac{\det \Delta^{(1)}}{\det \Delta} = \frac{1}{2\pi} - \frac{5}{2\pi^2} + \frac{4}{\pi^3} \tag{36}$$

An expansion of $\det \Delta^{(1)}$ into cyclic permutations forms a basis for the inclusion–exclusion series enumerating acyclic configurations containing C_1.

The next step is the introduction of loops canceling P_2 into the series. We define the matrix $\Delta^{(2)} = \Delta + \delta_{(2)}$, with the matrix $\delta_{(2)}$ converting to zero the following elements of $\Delta^{(2)}$: $[i_0, j']$, where j' is any n.n. of i_0 except j_4; elements $[j_2, i_0]$ and $[j_3, i_0]$. In addition, the matrix element $[j_1, i_0]$ becomes $-\varepsilon$ and diagonal elements become $[j_2, j_2] = [j_3, j_3] = 3$ and $[i_0, i_0] = 1$. Then the expression $\lim[\det \Delta^{(2)}/\varepsilon]$ as $\varepsilon \to \infty$ gives all configurations of arrows containing C_1 except the arrow directed from i_0 to j_2 and generating precisely one closed path of the type P_2 weighted with minus sign.

Although evaluation of det $\Delta^{(2)}$ is straightforward, the explicit form of matrices $G(r, r')$ and $\delta_{(2)}$ is worthy of notice. Taking rows and columns of these matrices in the natural order $i_0, j_1, j_2, j_3,\ldots,$ we can write the nonzero parts of the symmetric matrix G as

$$\begin{pmatrix} g_{0,0} & g_{0,0}-\frac{1}{4} & g_{0,0}-\frac{1}{4} & g_{0,0}-\frac{1}{4} \\ \cdots & g_{0,0} & g_{0,0}-1/\pi & g_{0,0}-1+2/\pi \\ \cdots & \cdots & g_{0,0} & g_{0,0}-1/\pi \\ \cdots & \cdots & \cdots & g_{0,0} \end{pmatrix} \tag{37}$$

and the matrix $\delta_{(2)}$ as

$$\begin{pmatrix} -3 & 1 & 1 & 1 \\ -\varepsilon & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \tag{38}$$

The matrix $\delta_{(2)}$ contains the nondiagonal element $-\varepsilon$, which tends to infinity at the close of evaluations. Because of this, any finite value of the diagonal element $[j_1, j_1]$ is irrelevant and we keep it unchanged. By the same reasoning, the Green function $g_{0,0}$ is no longer canceled in a matrix product $G\delta_{(2)}$.

Equation (24) with $\Delta^{(2)}$ leads to the expression

$$\lim_{\varepsilon\to\infty} \frac{\det \Delta^{(2)}}{\varepsilon \det \Delta} = \frac{1}{4\pi^2} - 4g_{0,0}\left(\frac{1}{2\pi^2} - \frac{1}{\pi^3}\right) \tag{39}$$

The Green function $g_{0,0}$ depends on the lattice size n and diverges as $\log(n)$ for large n. Therefore it must be canceled in further calculations.

The sum

$$S = \det \Delta^{(1)} + \lim_{\varepsilon\to\infty} \frac{\det \Delta^{(2)}}{\varepsilon} \tag{40}$$

is a part of the inclusion–exclusion series which enumerates arrow configurations containing C_1, generates neither P_1 nor P_2 separately, and, possibly, generates a combination of P_1 and P_2 having the form of a Θ-graph. Each Θ-graph being excluded twice brings a minus sign.

The last step consists in enumeration of arrow configurations generating Θ-graphs. A Θ-graph is a subgraph of L containing sites of two types: sites j with $\deg(j) = 2$ and two sites i_0 and i_1 with $\deg(i_0) = \deg(i_1) = 3$. For the Θ-graph in Fig. 7 the site i_0 is surrounded by three sites j_1, j_2, j_4 and the site i_1 by the sites a, b, c. The second group of sites may be oriented arbitrarily with respect to the first one.

We can try to construct a Θ-graph as follows. For fixed positions of the point i_1 and its neighbors a, b, c we should define a generating function of arrow configurations which generates three paths π_1, π_2, π_3 starting at sites a, b, c and ending at sites j_2, j_0, j_4. The combination of paths π_1, π_2, π_3 is equivalent to a Θ-graph with inverted arrows on the bonds belonging to two of them, π_1 and π_3. A generating function of the (34) generates only closed paths having no endpoints. To overcome this difficulty, we add to the original square lattice L three "bridges," additional bonds connecting the sites a and j_2; c and j_4; b and i_0. Accordingly, we introduce the matrix $\Delta^{(3)} = \Delta + \delta^{(3)}$ with a perturbed matrix $\delta^{(3)}$ such that three new nonzero elements of $\Delta^{(3)}$ appear: $[j_2, a] = [j_4, c] = [i_0, b] = -\varepsilon$. As above, $[j_3, i_0] = 0$ and $[j_3, j_3] = 3$. Also, $\delta^{(3)}$ converts to zero the elements $[i_1, j']$, where j' is any n.n. site of i_1 except b and reduces by 3 the diagonal element $[i_1, i_1]$. Then, applying the formula (34) to the new lattice L', we find that the expression $\lim[\det \Delta^{(3)}/\varepsilon^3]$ as $\varepsilon \to \infty$ gives all possible configurations of arrows on L' generating either three closed paths of the type $j_2 a \cdots j_2$, $i_0 b \cdots j_1 i_0$, $j_4 c \cdots j_4$ or a single path of type $j_2 a \cdots j_1 i_0 b \cdots j_4 c \cdots j_2$ or of the type $j_2 a \cdots j_4 c \cdots j_1 i_0 b \cdots j_2$. In both cases the arrows of closed paths belonging to the lattice L form the paths π_1, π_2, π_3 and, therefore, the desirable Θ-graph (with minus sign). Summation over all possible positions of the site i_1 and its three nearest neighbors gives the necessary improvement of the inclusion–exclusion expansion.

It is important to note that introducing the bridges imposes a purely topological character on our problem. To control the sign of the contribution coming from the Θ-graph, we must assure ourselves that the each path representation has a fixed parity. Generally, one, two, or three closed paths passing via three bridges are possible. But in our specific situation, only odd numbers of paths exist when the points i_0, j_2, and j_4 are neighbor sites as well as the points a, b, c and, in addition, the direction from the point j_3 to i_0 is forbidden. This is not the case for three-dimensional lattices and, therefore, our solution is restricted to planar lattices.

To reduce the technical problems arising from summation over all possible positions and mutual orientations of points i_1, a, b, c we introduce two different matrices $\Delta_i(L)$ and $\Delta_i(\Gamma)$ instead of $\Delta^{(3)}$, where the index i denotes the position of the point $i_1 = (k, l)$, and letters L and Γ reflect the positions of two neighbor sites a and b with respect to i_1 (Figs. 8a and 8b). The connection point of three lines in a Θ-graph on the square lattice has everywhere the form of the letter T. So, the letters L and Γ and their horizontal reflections can be embedded into the Θ-graph twice. Due to the left–right symmetry, the simple summation over i_1 is equivalent now to

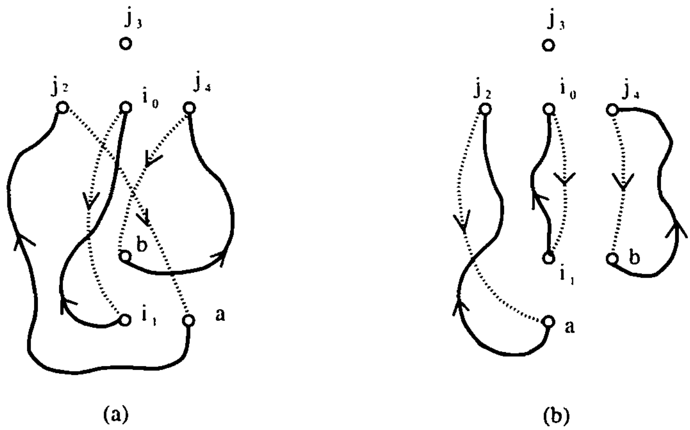

Fig. 8. Sites and bonds contributing to the definitions of the perturbed matrices (a) $\Delta_i(L)$ and (b) $\Delta_i(\Gamma)$. The broken lines mark the positions of bridges.

the summation of det $\Delta^{(3)}$ over i_1 and all orientations of its neighbors. The matrices $\Delta_i(L)$ and $\Delta_i(\Gamma)$ differ from Δ by the elements $[i_0, i_1] = [j_2, a] = [j_4, b] = -\varepsilon$ and by the elements $[j_3, i_0] = 0$ and $[j_3, j_3] = 3$.

The explicit form of the determinants for a given point $i_1 = (k, l)$ follows directly from Eq. (24):

$$\lim_{\varepsilon \to \infty} \frac{1}{\varepsilon^3} \left\{ \frac{\det \Delta_{i_1}(L)}{\det \Delta} + \frac{\det \Delta_{i_1}(\Gamma)}{\det \Delta} \right\} = \det M_L - \det M_\Gamma \tag{41}$$

where M_Γ and M_L are the matrices

$$\begin{pmatrix} g_{1,0} - g_{0,0} & g_{k,l} & g_{k+1,l} & g_{k,l-1} \\ g_{0,0} - g_{1,0} - 1 & g_{k,l-1} & g_{k+1,l-1} & g_{k,l-2} \\ g_{1,1} - g_{1,0} & g_{k-1,l} & g_{k,l} & g_{k-1,l-1} \\ g_{1,1} - g_{1,0} & g_{k+1,l} & g_{k+2,l} & g_{k+1,l-1} \end{pmatrix} \tag{42}$$

and

$$\begin{pmatrix} g_{1,0} - g_{0,0} & g_{k,l} & g_{k+1,l} & g_{k,l+1} \\ g_{0,0} - g_{1,0} - 1 & g_{k,l-1} & g_{k+1,l-1} & g_{k,l} \\ g_{1,1} - g_{1,0} & g_{k-1,l} & g_{k,l} & g_{k-1,l+1} \\ g_{1,1} - g_{1,0} & g_{k+1,l} & g_{k+2,l} & g_{k+1,l+1} \end{pmatrix} \tag{43}$$

Now we are going to take the sum of Eq. (41) over all $i_1 \in L$. In doing this we meet several problems.

First, we must take into account situations where the site i_1 is a nearest neighbor of i_0 or coincides with it. If either $i_0 = i_1$, $j_2 = a$, or $j_4 = b$, we set the corresponding diagonal elements of the matrices $\Delta_i(L)$ and $\Delta_i(\Gamma)$ to $-\varepsilon$: $[i_0, i_0] = -\varepsilon$, $[j_2, j_2] = -\varepsilon$, or $[j_4, j_4] = -\varepsilon$. The emergence of an infinitely large element of the diagonal when $\varepsilon \to \infty$ implies simply exclusion of the given point out of the graph. Equivalently, we can consider this element as an elementary closed loop starting and ending immediately at a given point and bringing the minus sign.

Second, if the sites i_0 and i_1 are nearest neighbors or coincide, it is not always easy to recognize a Θ-graph in the path representation. To this end, we show in Fig. 9 all these cases for the matrix $\Delta_i(L)$ providing them with examples of Θ-graphs. It is notable that the number of closed paths remains odd everywhere: one closed path in the cases of Figs. 9a, 9b, 9d, and 9e, and three paths, including an elementary loop, in the cases of Figs. 9c and 9f. So, all these nonstandard situations give Θ-graphs with correct sign. The matrix $\Delta_i(\Gamma)$ has similar properties.

Third, some of the arrangements of the site i_1 are forbidden, that is, no Θ-graphs correspond to them. For the matrix $\Delta_i(L)$ the forbidden positions of i_1 are i_0 and j_2, for the matrix $\Delta_i(\Gamma)$ these are j_2, i_0, j_3. The first pair is shown in Fig 10a and 10b. The definition of the perturbed matrices in these cases does not differ from the preceding cases and we must subtract them simply from the sum over $i_1 \in L$.

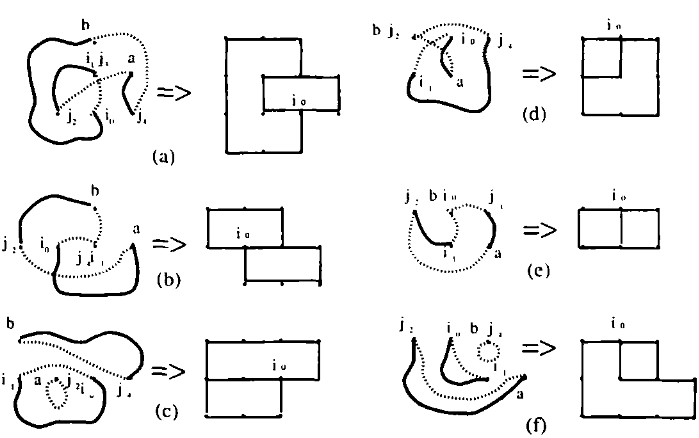

Fig. 9. The configurations of paths generated by the matrix $\Delta_{i_1}(L)$ when (a) $i_1 = j_3$; (b) $i_1 = j_4$; (c) $a = j_2$; (d) $b = j_2$; (e) $b = i_0$; (f) $b = j_4$.

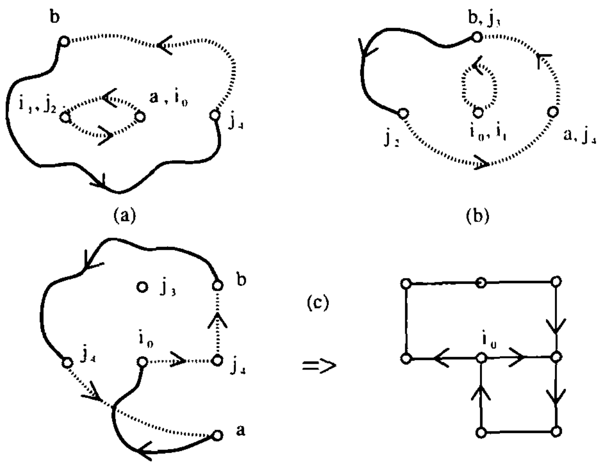

Fig. 10. The configurations of paths corresponding to forbidden arrangements of the site i_1: (a) $i_1 = j_2$; (b) $i_1 = i_0$; (c) Sites and bonds contributing to the definition of the matrix $\Delta(T)$.

Finally, it can be seen from Fig. 9 that each Θ-graph either is counted once as well as its horizontally symmetrical counterpart or is counted twice, whereas its counterpart has no representation by $\Delta_i(L)$ or $\Delta_i(\Gamma)$. The only exception is the case $\Delta_i(L)$ for $i_1 = j_4$ or, equivalently, Δ_{j_4} shown in Fig. 9b. Its counterpart has no representation and therefore we must take it twice. Similarly, we must take twice a contribution from $\Delta_{j_4}(\Gamma)$. Furthermore, the Θ-graph shown in Fig. 10c, right, has no representation either by $\Delta_i(L)$ or by $\Delta_i(\Gamma)$. Thus, we define the last perturbed matrix, denoting it by $\Delta(T)$, with the elements $[i_0, j_4] = [j_2, a] = [j_4, b] = -\varepsilon$ and $[j_3, i_0] = 0$, $[j_3, j_3] = 3$, where the new positions of points a and b are shown in Fig. 10c.

Gathering all these corrections, we may write the expression for the number of configurations generating a Θ-graph:

$$\begin{aligned} N(\Theta) = -\lim_{\varepsilon \to \infty} \frac{1}{\varepsilon^3} \Big\{ & \sum_{i_1} \det \Delta_{i_1}(L) + \sum_{i_1} \det \Delta_{i_1}(\Gamma) - \det \Delta_{j_2}(L) - \det \Delta_{i_0}(L) \\ & - \det \Delta_{j_2}(\Gamma) - \det \Delta_{i_0}(\Gamma) - \det \Delta_{j_3}(\Gamma) + \det \Delta_{j_4}(L) \\ & + \det \Delta_{j_4}(\Gamma) + 2 \det \Delta(T) \Big\} \end{aligned} \tag{44}$$

In order to evaluate two sums over i_1, notice that matrices (39) and (40) coincide except for the last columns. Combining them, we get

$$
\begin{aligned}
-&\det M_L+\det M_r \\
&=\frac{1}{16}\sum_{k,l}\frac{1}{(2\pi)^6}\int_0^{2\pi}\cdots\int\frac{i\sin(\beta_1)\det(M_1)\,e^{ik(\alpha_1+\alpha_2+\alpha_3)}e^{il(\beta_1+\beta_2+\beta_3)}}{D(\alpha_1,\beta_1)\,D(\alpha_2,\beta_2)\,D(\alpha_3,\beta_3)}\,d\alpha_1\cdots d\beta_3 \\
&\quad+\frac{1}{8}\sum_{k,l}\frac{1}{(2\pi)^4}\int_0^{2\pi}\cdots\int\frac{i\sin(\beta_1)\det(M_1')\,e^{ik(\alpha_1+\alpha_2)}e^{il(\beta_1+\beta_2)}}{D(\alpha_1,\beta_1)\,D(\alpha_2,\beta_2)}\,d\alpha_1\cdots d\beta_2
\end{aligned}
\tag{45}
$$

where the matrix M_1 is given by Eq. (8) with substitutions $\alpha_3=\alpha_1+\alpha_2$ and $\beta_3=\beta_1+\beta_2$, and the matrix M_1' has the form

$$
M_1'=\begin{pmatrix}
1 & g_{0,0} & e^{i\alpha_2} & 1 \\
3 & g_{0,0} & e^{i(\alpha_2-\beta_2)} & e^{-i\beta_1} \\
4/\pi-1 & g_{0,0} & 1 & e^{-i\alpha_1} \\
4/\pi-1 & g_{0,0} & e^{2i\alpha_2} & e^{i\alpha_1}
\end{pmatrix}
\tag{46}
$$

Surprisingly, the second integral in Eq. (45) equals zero and thus no contribution to the coefficient of $g_{0,0}$ comes from two first terms of Eq. (44). The first integral gives I_1 in Eq. (6) after replacing the order of summation and integration and subsequent integration over α_3, β_3.

The remaining terms in Eq. (44) can be easily obtained using tabulated data for the Green functions $g_{1,0}$, $g_{1,1}$, $g_{2,0}$, $g_{2,1}$. For example, the nonzero part of the matrix $\Delta(T)-\Delta$ is

$$
\begin{pmatrix}
0 & 0 & 0 & -\varepsilon & 0 & 0 \\
0 & 0 & 0 & 0 & -\varepsilon & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\varepsilon
\end{pmatrix}
\tag{47}
$$

and the corresponding part of the matrix G is

$$
\begin{pmatrix}
g_{0,0} & g_{1,0} & g_{1,0} & g_{1,0} \\
g_{1,0} & g_{0,0} & g_{1,1} & g_{2,0} \\
g_{1,0} & g_{1,1} & g_{0,0} & g_{1,1} \\
g_{1,0} & g_{2,0} & g_{1,1} & g_{0,0} \\
g_{1,1} & g_{2,1} & g_{2,1} & g_{1,0} \\
g_{1,1} & g_{2,1} & g_{1,0} & g_{1,0}
\end{pmatrix}
\tag{48}
$$

where rows and columns are in the following order: i_0, j_2, j_3, j_4, a, b. Substituting the product of these matrices in Eq. (24), we obtain

$$\lim_{\varepsilon\to\infty}\frac{\det \Delta(T)}{\varepsilon\det\Delta}=\frac{3}{64}-\frac{1}{4\pi}-\frac{1}{4\pi^2}+\frac{3}{\pi^3}-\frac{4}{\pi^4}+g_{0,0}\left(-\frac{3}{8}+\frac{13}{4\pi}-\frac{9}{\pi^2}+\frac{8}{\pi^3}\right) \tag{49}$$

The combined value of all remaining terms is

$$\frac{1}{8}-\frac{7}{8\pi}+\frac{5}{4\pi^2}+4g_{0,0}\left(\frac{1}{2\pi^2}-\frac{1}{\pi^3}\right) \tag{50}$$

The coefficient of $g_{0,0}$ is a crucial check of our calculations. Closed loops giving $g_{0,0}$ in Eq. (39) and Θ-graphs giving it in Eq. (50) have different geometrical structures. The resulting cancellation testifies to the correctness of the operation of the inclusion–exclusion principle in our problem.

The final expression for the number of configurations generating a Θ-graph is

$$\frac{N(\Theta)}{N}=\frac{I_1}{16}+\frac{1}{8}-\frac{7}{8\pi}+\frac{5}{4\pi^2}+4g_{0,0}\left(\frac{1}{2\pi^2}-\frac{1}{\pi^3}\right) \tag{51}$$

Combining Eqs. (36), (39), and (51) and using the symmetry of diagrams in Fig. 2 and Fig. 6a, we get

$$X_1=12\left[\det\Delta^{(1)}+\lim_{\varepsilon\to\infty}\frac{\det\Delta^{(2)}}{\varepsilon}+N(\Theta)\right] \tag{52}$$

Substitution of Eqs. (52) and (28) into Eq. (14) gives Eq. (3) quoted in the Introduction.

To find $P(3)$ [and therefore $P(4)$], it is necessary to determine $X_2^{(1)}$ and $X_2^{(3)}$. The procedure is quite similar to the one described above. Evaluating related determinants and summing over all possible configurations of Θ-graphs, we obtain

$$\frac{X_2^{(1)}}{N}=1-\frac{8}{\pi}+\frac{28}{\pi^2}-\frac{32}{\pi^3}-I_1-\frac{I_2}{8} \tag{53}$$

and

$$\frac{X_2^{(3)}}{N}=-\frac{3}{2}+\frac{6}{\pi}+\frac{2}{\pi^2}-\frac{16}{\pi^3}-\frac{I_1}{2}-\frac{I_2}{16} \tag{54}$$

These equations together with Eq. (32), give due to Eqs. (21) and (15), the final results for $P(3)$ and $P(4)$.

7. DISCUSSION

In this paper we have determined the height probabilities in the Abelian sandpile model of SOC on the square lattice. The method used here can be easily extended to other two-dimensional lattices, although the three-dimensional problem seems much more difficult.

The presented solution has some peculiarities in comparison with the known methods used for 2D lattice models of statistical mechanics. Exactly solved models can be divided conventionally into two classes. The first is the class of free-fermion models,[20] solutions of which usually have the form of a determinant or a Pfaffian. The second may be termed the class of interacting fermions and involves the Bethe-ansatz technique or the method of commuting transfer matrices.[21] In the first case, fermion variables in the space representation are free and in the second, fermions interact uniformly at each lattice site. The Abelian sandpile model takes an intermediate place between these cases. On one hand, the problem of enumerating all possible sandpile configurations in the SOC state is purely a free-fermion one. The generating function (34) underlying the solution can be represented as a product over independent closed paths (fermion trajectories) weighted with the minus sign. On the other hand, evaluating the height probabilities at a given lattice site leads to a problem containing an effective interaction between "fermions" at that site. In order to take into account this interaction, we have to introduce the Θ-graphs into consideration, which are an analog of loop diagrams in quantum field theory. The complexity of the diagrams depends on the order of the height–height correlation function. Thus, the Abelian sandpile is the source of a new class of lattice models which are characterized by a nontrivial space-dependent interaction.

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